Large energy solutions of the equivariant wave map problem

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A brief introduction to wave maps:

- Definition: Formally, wave maps are critical points of the Lagrangian

$$\mathcal{L}(u, \partial u) = \int_{\mathbb{R}^{1+d}} \eta^{\alpha\beta} \left\langle \partial_\alpha u, \partial_\beta u \right\rangle_g \, dt \, dx$$

where $u : (\mathbb{R}^{1+d}, \eta) \to (M, g)$. Here, η is the Minkowski metric on $\mathbb{R}^{1+d}$ and (M, g) is a Riemannian manifold.

- Intrinsic Formulation: Critical points of $\mathcal{L}$ satisfy the Euler-Lagrange equation

$$\eta^{\alpha\beta} D_\alpha \partial_\beta u = 0$$

- Extrinsic Formulation: If $M \hookrightarrow \mathbb{R}^N$ is embedded, critical points are characterized by

$$\Box u \perp T_u M$$

Cauchy problem:

- Intrinsic Formulation: In local coordinates on (M, g), the Cauchy problem for wave maps is

$$\Box u^k = -\eta^{\alpha\beta}\Gamma^k_{ij}(u)\partial_\alpha u^i \partial_\beta u^j$$
$$(u, \partial_t u)|_{t=0} = (u_0, u_1)$$

  where Γ^k_{ij} are the Christoffel symbols on TM.

- Extrinsic Formulation: In the embedded case, the Cauchy problem becomes

$$\Box u = \eta^{\alpha\beta} S(u)(\partial_\alpha u, \partial_\beta u)$$
$$(u, \partial_t u)|_{t=0} = (u_0, u_1)$$

  where S is the second fundamental form of the embedding.

Energy conservation and scaling

- Conservation of energy: Wave maps exhibit a conserved energy

$$E(u, \partial_t u)(t) = \int_{\mathbb{R}^d} (|\partial_t u|_g^2 + |\nabla u|_g^2)\, dx = \mathrm{const.}$$

- Scaling invariance: Wave maps are invariant under the scaling $u(t, x) \mapsto u(\lambda t, \lambda x)$. The conserved energy is invariant under the scaling $u(t, x) \mapsto \lambda^{\frac{d-2}{2}} u(\lambda t, \lambda x)$.
- Criticality: The scaling invariance implies that the Cauchy problem is $\dot{H}^s \times \dot{H}^{s-1}$ critical for $s = \frac{d}{2}$, energy critical when $d = 2$ and energy supercritical for $d > 2$.

Equivariant Wave Maps

Equivariant wave maps: In the presence of symmetries, e.g., $M = \mathbb{S}^d$, one can require

$$u \circ \rho = \rho^{\ell} \circ u$$

where $\rho \in SO(d)$ acts on $\mathbb{R}^d$ (resp. $\mathbb{S}^d$) by rotation. The action on $\mathbb{S}^d$ is rotation is about a fixed axis.

Foundational works in equivariant setting:

- Shatah (1988): finite time blow-up (self-similar) for wave maps $u : \mathbb{R}^{1+d} \to \mathbb{S}^d$ for $d \geq 3$.
- Shatah, Tahvildar-Zadeh (1992, 1994)

Small vs. Large Data: energy critical wave maps

- Small Data Theory–non-equivariant: If data (u_0, u_1) are small, then dynamics are simple, all solutions exists globally and become asymptotically free–referred to as scattering.
  - Tao ('01): for data small in critical Sobolev norm, $\dot{H}^1 \times L^2$, $d = 2$, maps to $\mathbb{S}^2$.
  - Krieger ('04): small energy data: maps to $\mathbb{H}^2$
  - Tataru ('05): small energy data: large class of targets M. Continuous dependence, persistence of regularity.
- Large data theory–non-equivariant Dynamical structure here is rich, as finite time breakdown (blow-up) may occur. Geometry of target plays a decisive role.
  - Negatively Curved Targets, e.g. $\mathbb{H}^2$: Correspond roughly to defocusing type equation. Global existence and scattering for all smooth energy data established in a remarkable series of papers, Krieger-Schlag ('09), Sterbenz-Tataru ('09), Tao (09).
  - Sterbenz-Tataru ('09): blow-up can occur only if target manifold admits nontrivial harmonic map. This type of result was previously seen by Struwe ('03) in equivariant setting.

$2d$ equivariant wave maps to the 2-sphere

Large data theory in case target does admit a nontrivial finite energy harmonic map:

- Sterbenz-Tataru ('09) proved threshold conjecture, i.e., g.e. and scattering for smooth data with energy below energy of harmonic map–non-equivariant.
- Many questions remain. Can one classify possible dynamics above threshold? Start with simpler equivariant model.

Model to be discussed today:

- $2d$ energy critical, equivariant wave maps:

$$u : \mathbb{R}^{1+2} \to \mathbb{S}^2$$

- equivariant reduction given by the ansatz

$$\begin{aligned} u(t, r, \omega) &= (\psi(t, r), \ell\omega) \\ &\mapsto (\sin\psi(t, r)\cos(\ell\omega), \sin\psi(t, r)\sin(\ell\omega), \cos\psi(t, r)) \end{aligned}$$

where (r, ω) are polar coordinates on $\mathbb{R}^2$ and $\ell \in \mathbb{N}^*$ is the equivariance class.

Energy Critical Cauchy Problem: In the 1-equivariant setting the Cauchy problem reduces to

$$\psi_{tt} - \psi_{rr} - \frac{1}{r}\psi_r + \frac{\sin(2\psi)}{2r^2} = 0 \qquad (1)$$
$$(\psi, \psi_t)|_{t=0} = (\psi_0, \psi_1)$$

Conserved Energy:

$$\mathcal{E}(\vec{\psi})(t) = \int_0^\infty \left(\psi_t^2 + \psi_r^2 + \frac{\sin^2(\psi)}{r^2}\right) r\,dr = \text{const.} \qquad (2)$$

$\Longrightarrow \psi(t,0) = m\pi\,,\ \psi(t,\infty) = n\pi$ for some $m, n \in \mathbb{N}$.

Disjoint energy classes–topological degree:

$$\mathcal{H}_{m,n} := \{(\psi_0, \psi_1) \,|\, \mathcal{E}(\psi_0, \psi_1) < \infty \text{ and } \psi_0(0) = m\pi, \psi_0(\infty) = n\pi\}.$$

Scaling Invariance: (1) and (2) are invariant under the scaling

$$(\psi(t,r), \dot{\psi}(t,r)) \mapsto (\psi(\lambda t, \lambda r)\lambda\dot{\psi}(\lambda t, \lambda r))$$

Equivariant wave maps to positively curved targets

In the case of positively curved targets, such as the sphere, 1-equivariant wave maps can blow-up in finite time.

- Struwe ('03): If finite time time blow-up occurs then $\Rightarrow$ bubbling off of finite energy harmonic sphere.
- Côte ('05): asymptotic instability of harmonic map.
- Krieger, Schlag, Tataru ('08), Raphael, Rodnianski ('09): Constructions of explicit blow-up solutions.
- Rodnianski, Sterbenz: Constructions for equivariance classes $\ell \geq 4$.

Role of the harmonic map

The role of the harmonic map is crucial in blow-up scenarios.
Harmonic Map: $Q(r) = 2\arctan(r)$, (stereographic projection).

- Equivariance and energy criticality imply blow up can only happen at the origin in an energy concentration scenario.
- Struwe ('03) showed that if blow up occurs at say $t = 1$, then, $\exists$ a seq. of times $t_n \to 1$ and $\exists$ a seq. of scales $\lambda_n \ll 1 - t_n$ so that the rescaled sequence of wave maps:

$$\vec{\psi}_n(t,r) = \left(\psi\left(t_n, \lambda_n r\right), \lambda_n \dot{\psi}\left(t_n, \lambda_n r\right)\right)$$

converges locally to $(Q(r/\lambda_0), 0)$ in the space-time norm $H^1_{\mathrm{loc}}((-1,1) \times \mathbb{R}^2; \mathbb{S}^2)$.

Topological degree and energy thresholds

Topological Degree: Finite energy $\Rightarrow \psi(t,0) = m\pi$, $\psi(t,\infty) = n\pi$ fixed for all t.

- Enough to look at , say $\psi(t,0) = 0$, $\psi(t,\infty) = n\pi$. Here we refer to n as the degree and we define

$$\mathcal{H}_n = \{(\psi_0, \psi_1) \mid \psi_0(0) = 0,\ \psi_0(\infty) = n\pi\}$$

- Note that $(Q, 0)$ is degree 1 since $Q(0) = 0$, $Q(\infty) = \pi$.
- Degree 0: CKM ('08): It is impossible for degree 0 data $\vec{\psi}(0) \in \mathcal{H}_0$ with $\mathcal{E}(\vec{\psi}) < 2\mathcal{E}(Q)$ to produce a bubble since such maps stay bounded away from south pole.

$$|\psi(r)| \leq F(\mathcal{E}(\vec{\psi})) < \pi \quad \forall \vec{\psi} \in \mathcal{H}_0 \quad \text{with} \quad \mathcal{E}(\vec{\psi}) < 2\mathcal{E}(Q)$$

- Degree 1: $(Q, 0)$ has minimal energy in $\mathcal{H}_1$ with $\mathcal{E}(Q, 0) = 4$. K-S-T ('08) produce finite time blow up for solutions in $\mathcal{H}_1$ with energy $\mathcal{E}(Q) + \delta$ for arbitrarily small $\delta > 0$.

Prior degree 0 results

First consider degree 0 maps, $\psi(t) \in \mathcal{H}_0$, i.e.,
$\psi(t,0) = \psi(t,\infty) = 0$.

- From Struwe ('03), one can deduce global existence for wave maps $\psi(t) \in \mathcal{H}_0$ with energy $\mathcal{E}(\vec{\psi}) < 2\mathcal{E}(Q)$.
- Côte, Kenig, Merle ('08) showed that degree 0 wave maps with energy slightly above the energy of Q, i.e., $\psi(t) \in \mathcal{H}_0$, $\mathcal{E}(\vec{\psi}) < \mathcal{E}(Q) + \delta$, scatter to a linear wave as $t \to \pm\infty$ for some small $\delta > 0$. They conjecture that scattering should hold for all degree 0 maps with energy $< 2\mathcal{E}(Q)$.
- One can actually deduce this conjecture from Sterbenz-Tataru ('09) by considering their results in the equivariant setting. Here we give another proof based on small data/concentration compactness/rigidity method of Kenig, Merle ('06), ('08).

Degree 0: GE and Scattering below sharp threshold

Theorem 1 (Sterbenz-Tataru ('09), Côte, Kenig, L., Schlag, ('12))

Let $\mathcal{E}(\vec{\psi}) < 2\mathcal{E}(Q,0)$, $\vec{\psi}(0) \in \mathcal{H}_0$. Then the solution exists globally, and scatters (energy on compact sets vanishes as $t \to \infty$). In other words, one has

$$\vec{\psi}(t) = \vec{\varphi}(t) + o_{\mathcal{H}}(1) \quad as \quad t \to \infty$$

where $\vec{\varphi} \in \mathcal{H}$ solves the linearized equation

$$\varphi_{tt} - \varphi_{rr} - \frac{1}{r}\varphi_r + \frac{1}{r^2}\varphi = 0$$

Moreover, this result is sharp in $\mathcal{H}_0$ in the following sense: For all $\delta > 0$ there exists data $\vec{\psi}(0) \in \mathcal{H}_0$ with $\mathcal{E}(\vec{\psi}) \leq 2\mathcal{E}(Q) + \delta$, such that $\vec{\psi}$ blows up in finite time.

Degree 1: Classification?

- Krieger, Schlag Tataru ('08): the blow up solutions exhibit a decomposition of the form

$$\psi(t,r) = Q(r/\lambda(t)) + \epsilon(t,r)$$

  where the concentration rate satisfies $\lambda(t) = (1-t)^{1+\nu}$ for $\nu > \frac{1}{2}$, and $\epsilon(t) \in \mathcal{H}_0$ is small and regular.
- Here we consider the converse problem. Namely, if blow-up does occur for a solution $\vec{\psi}(t) \in \mathcal{H}_1$, in which energy regime, and in what sense does such a decomposition always hold?
- Given degree 0 result it is natural to suspect universality of the above blow-up profile in energy regime $[\mathcal{E}(Q), 3\mathcal{E}(Q))$ since Struwe's bubbling off theorem suggests that the difference $\vec{\psi}(t,r) - (Q(r/\lambda(t)), 0) \in \mathcal{H}_0$ has energy $\approx \mathcal{E}(\vec{\psi}) - \mathcal{E}(Q)$ which is $< 2\mathcal{E}(Q)$ if $\mathcal{E}(\vec{\psi}) < 3\mathcal{E}(Q)$.

Degree 1: Finite time blow-up

Theorem 2 (Côte, Kenig, L., Schlag, ('12))

Let $\vec{\psi}(t) \in \mathcal{H}_1$ be a smooth solution blowing up at time $t = 1$ with

$$\mathcal{E}(\vec{\psi}) < 3\mathcal{E}(Q).$$

Then, there exists a continuous function, $\lambda : [0,1) \to (0,\infty)$ with $\lambda(t) = o(1-t)$, a map $\vec{\varphi} = (\varphi_0, \varphi_1) \in \mathcal{H}_0$ with $\mathcal{E}(\vec{\varphi}) = \mathcal{E}(\vec{\psi}) - \mathcal{E}(Q)$, and a decomposition

$$\vec{\psi}(t) = \vec{\varphi} + \left(Q\left(\cdot/\lambda(t)\right), 0\right) + \vec{\epsilon}(t)$$

such that $\vec{\epsilon}(t) \in \mathcal{H}_0$ and $\vec{\epsilon}(t) \to 0$ in $\mathcal{H}_0$ as $t \to 1$.

Remarks

- The techniques used to prove Theorem 2 were inspired by the recent works of Duyckaerts, Kenig, Merle who established analogous classification results for

$$\Box u = u^5$$

in $\dot{H}^1 \times L^2(\mathbb{R}^3)$ with $W(x) = (1 + |x|^2/3)^{-\frac{1}{2}}$ instead of Q.

- We use certain parts of their ideology, which is very heavily based on concentration compactness arguments. Note that here we cannot rely in any form on induction on energy.
- We also establish a similar classification for degree 1 global-in-time solutions. Note that here the topology prevents scattering to 0, but rather solutions can asymptotically decouple into a rescaled harmonic map plus pure radiation.

Degree 1: Global solutions

Theorem 3 (Côte, Kenig, L., Schlag, ('12))

Let $\vec{\psi}(t) \in \mathcal{H}_1$ be a smooth solution that exists globally in time with

$$\mathcal{E}(\vec{\psi}) < 3\mathcal{E}(Q).$$

Then, there exists a continuous function, $\lambda : [0, \infty) \to (0, \infty)$ with $\lambda(t) = o(t)$, a solution to the linearized equation $\vec{\varphi}_L(t) \in \mathcal{H}_0$, and a decomposition

$$\vec{\psi}(t) = \vec{\varphi}_L(t) + (Q\left(\cdot/\lambda(t)\right), 0) + \vec{\epsilon}(t)$$

such that $\vec{\epsilon}(t) \in \mathcal{H}_0$ and $\vec{\epsilon}(t) \to 0$ in $\mathcal{H}_0$ as $t \to \infty$.

Remarks

The requirement $\lambda(t) = o(t)$ as $t \to \infty$: many possibilities for the asymptotic behavior of global deg. 1 solutions w. energy $< 3\mathcal{E}(Q)$.

- scattering to Q_{λ_0}: If $\lambda(t) \to \lambda_0 \in (0, \infty)$ then our theorem says that the solution $\psi(t)$ asymptotically decouples into a soliton, Q_{λ_0}, plus a purely dispersive term.
- infinite time blow-up: If $\lambda(t) \to 0$ as $t \to \infty$ then the solution is concentrating $\mathcal{E}(Q)$ worth of energy at the origin as $t \to \infty$.
- infinite time flattening: If $\lambda(t) \to \infty$ as $t \to \infty$ then the solution concentrates $\mathcal{E}(Q)$ worth of energy at spacial infinity as $t \to \infty$.
- Global solutions of the type mentioned above, i.e., infinite time blow-up and flattening, have been constructed in the case of the $3d$ semi-linear focusing energy critical wave equation $\Box u = u^5$ by Donninger, Krieger ('11). No constructions of this type are known at this point for energy critical wave maps.

Further Remarks

- Theorem 2 and 3 together give a classification of all possible dynamics for deg. 1 maps in the energy regime $[\mathcal{E}(Q), 3\mathcal{E}(Q))$.
- Of course, our results do not give information about the precise rates $\lambda(t)$.
- We also say nothing about what happens at thresholds or above, i.e., $\mathcal{E} \geq 2\mathcal{E}(Q)$ in deg. 0 case and $\mathcal{E} \geq 3\mathcal{E}(Q)$ for the deg. 1 classification results.
- It is possible that at these higher energies one has more complicated dynamics such as multiple bubbles forming. As of yet no such multi-bump solutions have been constructed. Similarly, no multi-bump solutions have been constructed for $\Box u = u^5$.

Proof of Degree 0 Theorem: Induction on Energy

Kenig-Merle method: We outline the proof of Theorem 1. Let

$$\mathcal{S} = \{(\psi_0, \psi_1) \in \mathcal{H}_0 \,|\, \vec{\psi}(t) \,\text{exists globally and scatters as}\, t \to \pm\infty\}$$

We claim that $\mathcal{E}(\vec{\psi}) < 2\mathcal{E}(Q) \Rightarrow \vec{\psi} \in \mathcal{S}$.

- (Small data result): Small data global existence and scattering, proving $\mathcal{S}$ is not empty.
- (Concentration Compactness): If theorem fails, then $\exists$ nonzero energy solution $\vec{\psi}_*$ of minimal energy $\mathcal{E}^* < 2\mathcal{E}(Q)$ which does not scatter (called the critical element). $\exists A_0 > 0$, and a continuous function $\lambda : I_{\max} \to [A_0, \infty)$ s.t. the set

$$K := \left\{ \left(\psi_* \left(t, r/\lambda(t)\right), \, \lambda^{-1}(t)\dot{\psi}_* \left(t, r/\lambda(t)\right) \right) \,\Big|\, t \in I_{\max} \right\}$$

  is pre-compact in $\mathcal{H}_0$. (Bahouri-Gerard Concentration Compactness decomposition ('99).)
- (Rigidity Argument): If a global evolution $\vec{\psi}$ has the property that the trajectory, K, is pre-compact in $\mathcal{H}_0$, then $\psi \equiv 0$.

Comments on Degree 0 maps

Passage to $4d$ semi-linear formulation: Strong repulsive potential term hidden in the nonlinearity:

$$\frac{\sin(2\psi)}{2r^2} = \frac{\psi}{r^2} + \frac{\sin(2\psi) - 2\psi}{2r^2} = \frac{\psi}{r^2} + \frac{O(\psi^3)}{r^2}$$

- Indicates that the linearized operator has more dispersion than the $2d$ wave. In fact, same dispersion as $4d$ wave.
- Setting $\psi = ru$ we are led to this equation for u:

$$u_{tt} - u_{rr} - \frac{3}{r}u_r + \frac{\sin(2ru) - 2ru}{2r^3} = 0$$

- The nonlinearity above has the form $N(u, r) = u^3 Z(ru)$, Z smooth, bounded, even. The linear part is the radial d'Alembertian in $\mathbb{R}^{1+4}$ and linearization is free radial wave equation in $\mathbb{R}^{1+4}$:

$$v_{tt} - v_{rr} - \frac{3}{r}v_r = 0.$$

4d reduction for degree zero wave map

Observe that for $\vec{\psi} \in \mathcal{H}_0$ we have that

$$\mathcal{E}(\vec{\psi}) \le \|\vec{\psi}\|^2_{H \times L^2} := \int_0^\infty \left(\psi_t^2 + \psi_r^2 + \frac{\psi^2}{r^2} \right) r dr = \int_0^\infty (u_t^2 + u_r^2)\, r^3 dr.$$

- If we assume that $\mathcal{E}(\vec{\psi}(0)) < 2\mathcal{E}(Q)$ then, we also have the opposite inequality

$$\|\vec{u}(0)\|^2_{\dot{H}^1 \times L^2} \lesssim \mathcal{E}(\vec{\psi}(0)).$$

- $2d$, degree 0 wave map problem equivalent to $4d$ cubic semi-linear. Moreover, a sequence of $2d$ degree 0 maps with energy bounded below $2\mathcal{E}(Q)$ correspond to a uniformly bounded sequence in $\dot{H}^1 \times L^2(\mathbb{R}^4)$.
- This correspondence below $2\mathcal{E}(Q)$ means we can use technology for $4d$ equations, in particular Bahouri-Gérard concentration concentration compactness procedure, and new exterior enegy estimates for free radial wave of Côte, Kenig, Schlag.

Bahouri-Gérard Decomposition

Bahouri-Gerard Decomposition

$\{\vec{\psi}_n\} \subset \mathcal{H}_0$ seq. bounded $< 2\mathcal{E}(Q)$. Then, up to extracting a subsequence, $\exists$ a seq. of linear waves $\vec{\varphi}_L^j \in \mathcal{H}_0$, a seq. of times $\{t_n^j\}$, a seq. of scales $\{\lambda_n^j\} \subset (0,\infty)$, s.t. for $\vec{\gamma}_n^k$ defined by

$$\vec{\psi}_n(r) = \sum_{j=1}^{k} \left(\varphi_L^j(-t_n^j/\lambda_n^j, r/\lambda_n^j), \frac{1}{\lambda_n^j} \dot{\varphi}_L^j(-t_n^j/\lambda_n^j, r/\lambda_n^j) \right) + \vec{\gamma}_n^k(r)$$

we have, for any $j \le k$, that

$$(\gamma_n^k(\lambda_n^j t_n^j, \lambda_n^j \cdot), \lambda_n^j \gamma_n^k(\lambda_n^j t_n^j, \lambda_n^j \cdot)) \rightharpoonup 0 \quad \textit{weakly in} \quad H \times L^2.$$

In addition, for any $j \neq k$ we have

$$\frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + \frac{\left|t_n^j - t_n^k\right|}{\lambda_n^j} + \frac{\left|t_n^j - t_n^k\right|}{\lambda_n^k} \to \infty \quad \textit{as} \quad n \to \infty.$$

Bahouri-Gérard Decomposition Cont.

Bahouri-Gerard Decomposition cont.

Moreover, the errors $\vec{\gamma}_n^k$ *vanish asymptotically in the sense that if we let* $\gamma_{n,L}^k(t) \in \mathcal{H}_0$ *denote the linear evolution of the data* $\vec{\gamma}_n^k \in \mathcal{H}_0$, *we have*

$$\limsup_{n\to\infty} \left\| \frac{1}{r}\gamma_{n,L}^k \right\|_{L_t^\infty L_x^4 \cap L_t^3 L_x^6(\mathbb{R}\times\mathbb{R}^4)} \to 0 \quad as \quad k \to \infty.$$

Finally, we have the almost-orthogonality of the $H \times L^2$ *norms :*

$$\|\vec{\psi}_n\|_{H\times L^2}^2 = \sum_{1\le j\le k} \|\vec{\varphi}_L^j(-t_n^j/\lambda_n^j)\|_{H\times L^2}^2 + \|\vec{\gamma}_n^k\|_{H\times L^2}^2 + o_n(1)$$

and the almost-orthogonality of the nonlinear energy:

$$\mathcal{E}(\vec{\psi}_n) = \sum_{j=1}^k \mathcal{E}(\vec{\varphi}_L^j(-t_n^j/\lambda_n^j)) + \mathcal{E}(\vec{\gamma}_n^k) + o_n(1)$$

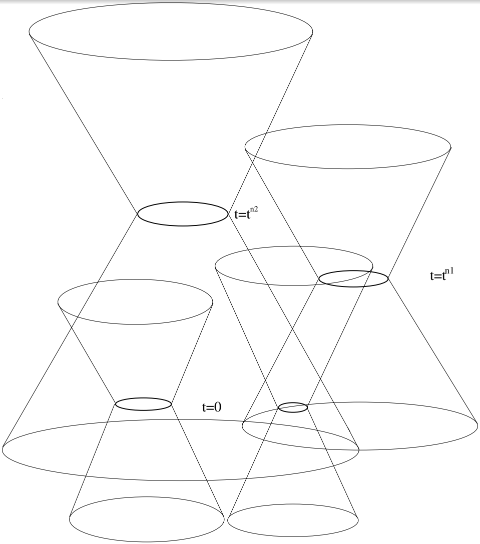

Figure : a schematic description of the concentration-compactness decomposition

Classification: how to work with degree 1 maps

- Degree 0 wave maps with energy below $2\mathcal{E}(Q)$ are analytically tractable objects given correspondence with $4d$ semi-linear equation.
- Nontrivial geometry of degree 1 wave maps is an obstacle to such simplifications.
- We rely explicitly on classical results on equivariant wave maps from the 90's and early 00's to bridge divide between degree 0 maps, on which can use concentration compactness techniques, and degree 1 maps, which we wish to classify.
- I will outline our procedure for proving Theorem 2 – our classification of finite time blow-up. The general outline for proving Theorem 3 is similar in spirit.

Classical results

- Shatah, Tahvildar-Zadeh ('92), Exterior energy decay:

$$\forall 0 \leq \lambda < 1 \quad \mathcal{E}^{1-t}_{\lambda(1-t)}(\vec{\psi}(t)) \to 0 \quad \text{as} \quad t \to 1$$

- Shatah, Tahvildar-Zadeh ('92), vanishing of averaged kinetic energy:

$$\frac{1}{1-t}\int_t^1 \int_0^{1-s} \dot{\psi}^2(s,r)\, r\, dr\, ds \to 0 \quad \text{as} \quad t \to 1$$

- Struwe's bubbling off theorem ('03): If $\mathcal{E}(\vec{\psi}) < 3\mathcal{E}(Q)$ then Struwes theorem implies that $\exists$ a seq. of times $\{t_n\}$, a sequence of scales λ_n with $\lambda_n \ll 1 - t_n$ so that

$$\psi(t_n + \lambda_n t, \lambda_n r) - Q(r) \to 0 \quad \text{in} \quad L^2((-1,1); H_{\mathrm{loc}})$$

Extraction of the large profile, Q_{λ_n}

- Using the classical results, we can (passing to a subsequence and rescaling) find $\alpha_n \to \infty$ so that

$$\|\psi(t_n) - Q(\cdot/\lambda_n)\|^2_{H(r \le \alpha_n \lambda_n)} \to 0 \quad \text{as} \quad n \to \infty$$

$$\int_0^{1-t_n} \dot{\psi}^2(t_n, r)\, r\, dr \to 0 \quad \text{as} \quad n \to \infty$$

- Then, for any $0 < r_n < \alpha_n \lambda_n$ we have

$$\psi(t_n, r_n) \to \pi \quad \text{as} \quad n \to \infty$$

$$\mathcal{E}^{\infty}_{r_n}(\vec{\psi}(t_n) - (Q(\cdot/\lambda_n), 0)) \le C \le 2\mathcal{E}(Q)$$

Extraction of the radiation term

- Outside the light cone, a blow-up solution remains smooth up to $t = 1$. We seek to isolate the singular part of the wave map by extracting the regular part of the solution outside of the light cone.
- This is accomplished by taking a limit after chopping off nontrivial topology of $\psi(t_n)$ at points $r_n < 1 - t_n$. Idea is to construct degree 0 maps $\varphi_n \in \mathcal{H}_0$ that agree with $\vec{\psi}(t_n) - (\pi, 0)$ on the interval $[r_n, \infty)$.

$$\vec{\varphi}_n \to \vec{\varphi} \quad \text{in} \quad H \times L^2$$

deg. 1 becomes deg. 0

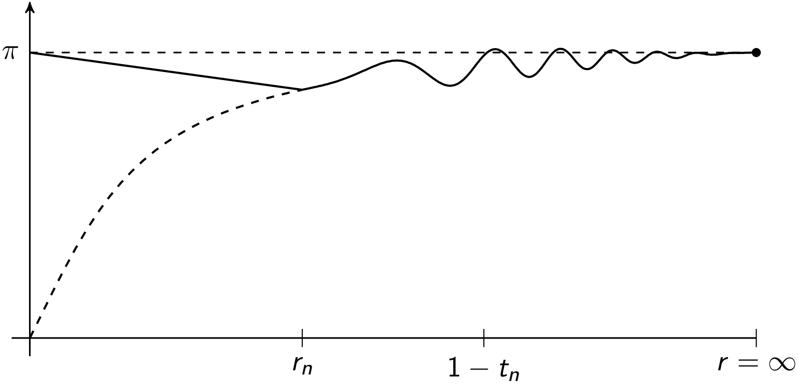

Figure : The solid line represents the graph of the function $\varphi_n + \pi$ for fixed n, described in previous slide. The dotted line is the piece of the function $\psi(t_n, \cdot)$ that is chopped at $r = r_n$ in order to linearly connect to π, which ensures that $\vec{\varphi}_n \in \mathcal{H}_0$.

Singular part $\vec{\psi}(t) - \vec{\varphi}(t)$ and the error, $\vec{\epsilon}_n$

- Now consider the backwards wave map evolution $\vec{\varphi}(t) \in \mathcal{H}_0$ of the limit $\vec{\varphi}_n$. This is degree 0, and satisfies $\mathcal{E}(\vec{\varphi}) < 2\mathcal{E}(Q)$ so the evolution is global, smooth, and scatters.
- By the finite speed of propagation $\psi(t,r) - \varphi(t,r) = \pi$ for $r \in [1-t, \infty)$ and $t \in [0,1)$.
- Now that we have identified the blow-up profile Q_{λ_n} along a time sequence and the radiation term, $\varphi(t)$ we can examine what is left

$$\vec{\epsilon}_n = (\epsilon_n^0, \epsilon_n^1) := \vec{\psi}(t_n) - \vec{\varphi}(t_n) - (Q(\cdot/\lambda_n), 0)$$

- First note: $\vec{\epsilon}_n \in \mathcal{H}_0$. Can also show that $\mathcal{E}(\vec{\epsilon}_n) \leq C \leq 2\mathcal{E}(Q)$ and

$$\|\epsilon_n^1\|_{L^2} \to 0 \quad \text{as} \quad n \to \infty$$

Compactness of the error $\vec{\epsilon}_n$

- To finish the proof (along a sequence of times), we need to show that

$$\|\epsilon_n\|_H^2 := \int_0^\infty \left(\partial_r \epsilon_n^2(r) + \frac{\epsilon_n^2}{r^2} \right) r\, dr \to 0 \quad \text{as} \quad n \to \infty$$

- The proof is motivated by work of Duyckaerts, Kenig, Merle for $\Box u = u^5$ in $\mathbb{R}^{1+3}$. Delicate technical argument involving several steps. Concentration compactness techniques, our deg. 0 theory, and exterior energy estimates for $4d$ free waves of Côte, Kenig, Schlag ('12), are crucial.
- First define wave map evolutions $\vec{\epsilon}_n(t)$ of the data $\vec{\epsilon}_n \in \mathcal{H}_0$. Global and time and scatter since they are deg. 0 and have energy $< 2\mathcal{E}(Q)$.

Compactness of the error $\vec{\epsilon}_n$

Step 1 Show that the sequence $\vec{\epsilon}_n$, which is bounded in $H \times L^2$, contains no nonzero profiles.

- If it did, these profiles would necessarily be $= Q_{\lambda_0}$ due to vanishing of ϵ_n^1 in L^2
- This gives compactness in Strichartz norm $\|\frac{1}{r}\epsilon_n\|_{L_t^3 L_x^6(\mathbb{R}^4)} \to 0$, but not in energy.
- Important implication: $\exists$ linear waves $\vec{\epsilon}_{n,L}(t)$ with data having 0 initial velocities, $\vec{\epsilon}_{n,L}(0) = (\epsilon^0_{n,L}, 0)$ so that

$$\sup_{t\in\mathbb{R}} \|\vec{\epsilon}_n(t) - \vec{\epsilon}_{n,L}(t)\|_{(H\times L^2)} \to 0 \quad \text{as} \quad n \to \infty$$

- This allows us to use linear theory.

Exterior energy for $4d$ linear wave equation

In companion paper, Côte, Kenig, Schlag ('12) prove the following estimates for free radial wave in $\mathbb{R}^{1+4}$:

Theorem

Consider solution $v(t)$ to $\Box v = 0$, $\vec{v}(0) = (f, 0)$ in $\mathbb{R}^{1+4}$. Then, $\exists c > 0$ so that

$$\|v(t)\|_{\dot{H}^1 \times L^2(r \geq t)} \geq c\|f\|_{\dot{H}^1}$$

- The above estimates hold for data $(f, 0)$ in dimensions $d = 0 \mod 4$ but fail in dimensions $d = 2 \mod 4$.
- The corresponding estimate for data $(0, g)$ hold for $d = 2 \mod 4$ but fail for $d = 0 \mod 4$.

Use linear theory in compactness argument

Step 2 Use linear theory in a contradiction argument:

- If $\|\epsilon_n\|_H$ does not tend to zero, then up to a subsequence we have

$$\|\epsilon_n\|_H \geq \alpha_0$$

- Using the fact that the sequence of nonlinear evolutions contain no profiles together with $4d$ correspondence and linear theory we have lower bound for exterior energy of nonlinear evolution:

$$\|\vec{\epsilon}(t)\|_{H\times L^2(r\geq t)} \geq c\alpha_0$$

- Using concentration compactness techniques, one can show that evolutions of $\vec{\psi}(t_n)$ and the error $\vec{\epsilon}_n$ remain close on an interval and with the above estimates, this leads to a concentration of energy away from the origin at a time < 1 which is a contradiction.

The End

Thank you!

p.s. a version of the slides from this talk will be availably shortly on my webpage:

math.uchicago.edu/∼alawrie